

Power and non-power expansions of the solutions for the fourth-order analogue to the second Painlevé equation

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Abstract

Fourth - order analogue to the second Painlevé equation is studied. This equation has its origin in the modified Korteweg - de Vries equation of the fifth order when we look for its self - similar solution. All power and non - power expansions of the solutions for the fourth - order analogue to the second Painlevé equation near points $z = 0$ and $z = \infty$ are found by means of the power geometry method. The exponential additions to solutions of the equation studied are determined. Comparison of the expansions found with those of the six Painlevé equations confirm the conjecture that the fourth - order analogue to the second Painlevé equation defines new transcendental functions.

1 Introduction.

More than one century ago Painlevé and his collaborators analyzed a certain class of the second-order nonlinear ordinary differential equations (ODE). In fact Painlevé examined two problems. The first one was to find all the second-order canonical equations with the solutions without movable critical points. The second problem was to pick out those equations which have solutions defining new special functions. The former was posed by Picard in 1876. While the latter was mentioned by A. Poincaré and L. Fuchs in 1884.

As a result of investigations Painlevé and his school discovered six second-order nonlinear ODEs which solutions could not be expressed in terms of known

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elementary or special functions. Nowadays they go under the name of the Painlevé equations and their solutions are called the Painlevé transcendents.

For a long period of time the Painlevé equations were regarded as nothing more than a curious moment in the theory of differential equations and were beyond the center of scientific circles attention. However since the sixtieth of the twentieth century a number of works appeared where it was shown that the Painlevé transcendents arose in the models describing physical phenomena as frequently as many other special functions [1–9]. This fact caused a significant interest to the studying of their properties and set a problem to find other nonlinear ODEs defining new transcendents.

It is necessary to mention that exact solutions of many partial nonlinear differential equations (such as the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation and so on) solvable by the inverse scattering transform can be expressed via the Painlevé transcendents and it is one of their most important applications [1,5,8,10,29]. In this connection it is quiet natural to suppose that the partial solutions of exactly solvable equations of the order higher than the order of previously mentioned equations also can be expressed in terms of the transcendents defined as the solutions of nonlinear ODEs. This idea was developed in the work [10], where an hierarchy of the first Painlevé equation was introduced. Nowadays the higher analogues of the Painlevé equations are intensively studied [10–41].

Taking after [10,24,29,33] we will show that a solution of the modified Korteweg - de Vries equation of the fifth order

$$u_t + u_{xxxxx} - 10u^2u_{xxx} - 40u_xu_{xx} - 10u_x^3 + 30u^4u_x = 0 \quad (1.1)$$

is defined by the solution of the fourth-order analogue of the second Painlevé equation. Let us look for the solution of the equation (1.1) in the form

$$u(x, t) = \frac{1}{(5t)^{1/5}} w(z), \quad z = \frac{x}{(5t)^{1/5}}. \quad (1.2)$$

After substitution of (1.2) into (1.1) and integrating once by z we get the equation

$$f(z, w) \stackrel{def}{=} w_{zzzz} - 10w^2w_{zz} - 10ww_z^2 + 6w^5 - zw - \alpha = 0 \quad (1.3)$$

This equation has a number of properties similar to that of Painlevé equations. More exactly it possesses Bäcklund transformations [12,25,29], a Lax pair [12], rational and special solutions at certain values of the parameter α [12,18,25,29]. The Cauchy problems for the equation (1.3) can be solved by the isomonodromic transform method. It is likely that the equation (1.3) defines

new transcendental functions as Painlevé equations do. However the exact proof of this statement yet is an open question. In this connection an important step of the investigation of the equation (1.3) properties is the finding of all its solution asymptotics.

The aim of this work is to calculate all power and non-power asymptotics, power and exponential expansions for the solutions of the equation (1.3) with a help of the power geometry method [42–44].

The problem definition is

1) to find all power asymptotics of its solutions. If a solution of the equation supposing that $z \rightarrow 0$ or $z \rightarrow \infty$ can be presented in the form

$$w(z) = c_r z^r + o(|z|^{r+\varepsilon}), \quad (1.4)$$

where the coefficient $c_r = \text{const} \in \mathbb{C}$, $c_r \neq 0$, the exponents $r, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and

$$\omega = \begin{cases} -1, & z \rightarrow 0; \\ 1, & z \rightarrow \infty, \end{cases}$$

then the power asymptotic of the solution (1.4) is

$$w(z) = c_r z^r. \quad (1.5)$$

2) to calculate all power expansions of its solutions given by

$$w(z) = c_r z^r + \sum_s c_s z^s, \quad (1.6)$$

where $s > r$ if $z \rightarrow 0$ and $s < r$ if $z \rightarrow \infty$.

3) to find for the studied equation all non-power asymptotics, i.e. functions $u(z)$ that are connected with the solutions of the equation (1.3) in the way

$$w(z) = u(z) [1 + o(|z|^{\omega\varepsilon} |u(z)|^{\omega_1\varepsilon})] \quad (1.7)$$

for some $\varepsilon < 0$ where

$$\omega_1 = \begin{cases} 0, & u(z) \rightarrow \text{const} \neq 0; \\ -1, & u(z) \rightarrow 0; \\ 1, & u(z) \rightarrow \infty. \end{cases}$$

4) to find all exponential additions to power expansions of the solutions which correspond to the exponentially close solutions. In other words it is required to find functions

$$v(z) = \exp \left[\sum \gamma_s z^s \right], \quad \omega s < \text{const}, \quad (1.8)$$

which, added to the solution $w(z) + v(z)$, correspond to the power expansions.

This paper outline is as follows. The general properties of the fourth - order analogue to the second Painlevé equation are discussed in section 2. Power expansions corresponding to the apexes and to the edges are given in sections 3, 4, 5, 6 and 7. In section 8 we consider the general properties of the equation studied in the special case (at $\alpha = 0$). Corresponding power and non-power expansions of solutions are presented in sections 9, 10 and 11. The three-level exponential additions are determined in sections 12, 13, 14, 15, 16, 17 and 18. The results of the work are gathered in section 19.

2 General properties of the equation (1.3) at $\alpha \neq 0$.

The following points correspond to the monomials of the studied equation: $M_1 = (-4, 1)$, $M_2 = (-2, 3)$, $M_3 = (-2, 3)$, $M_4 = (0, 5)$, $M_5 = (1, 1)$, $M_6 = (0, 0)$. The carrier $S(f)$ of the equation contains five points $Q_1 = M_1$, $Q_2 = M_4$, $Q_3 = M_5$, $Q_4 = M_6$, $Q_5 = M_2 = M_3$. Their convex hull is the quadrangle with four apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) and four edges $\Gamma_1^{(1)} = [Q_1, Q_2]$, $\Gamma_2^{(1)} = [Q_2, Q_3]$, $\Gamma_3^{(1)} = [Q_3, Q_4]$, $\Gamma_4^{(1)} = [Q_4, Q_1]$ (fig. 1).

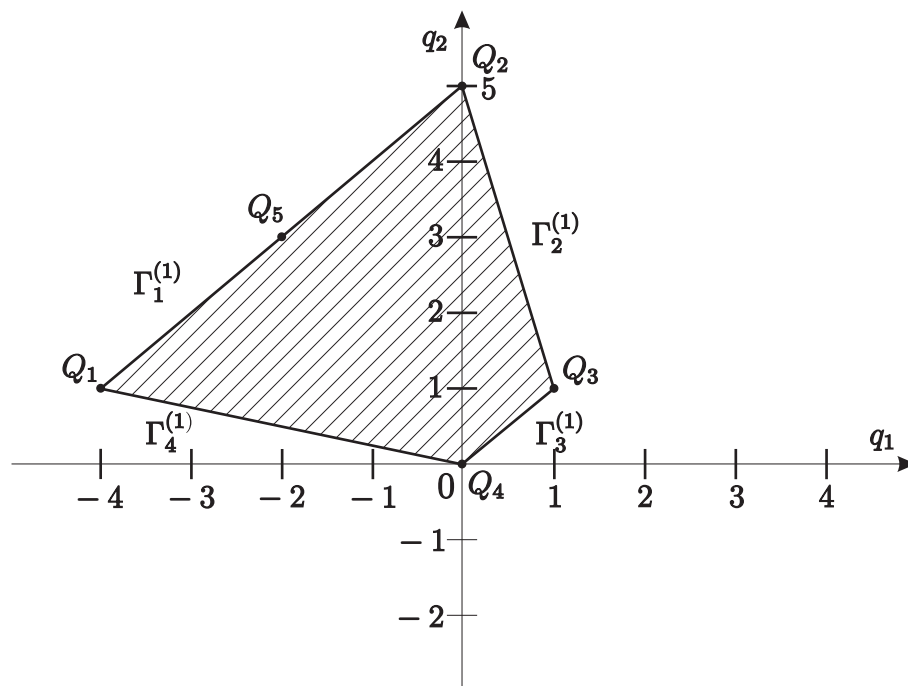


Fig. 1.

The external normal vectors N_j ($j = 1, 2, 3, 4$) to edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3, 4$) are $N_1 = (-1, 1)$, $N_2 = (4, 1)$, $N_3 = (1, -1)$, $N_4 = (-1, -4)$. They form the

normal cones $U_j^{(1)}$ of edges $\Gamma_j^{(1)}$

$$U_j^{(1)} = \{P = \lambda N_j, \lambda > 0\}, \quad j = 1, 2, 3, 4. \quad (2.1)$$

The normal cones $U_j^{(0)}$ of the apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) are the angles between the edges that adjoin to the apex (fig. 2).

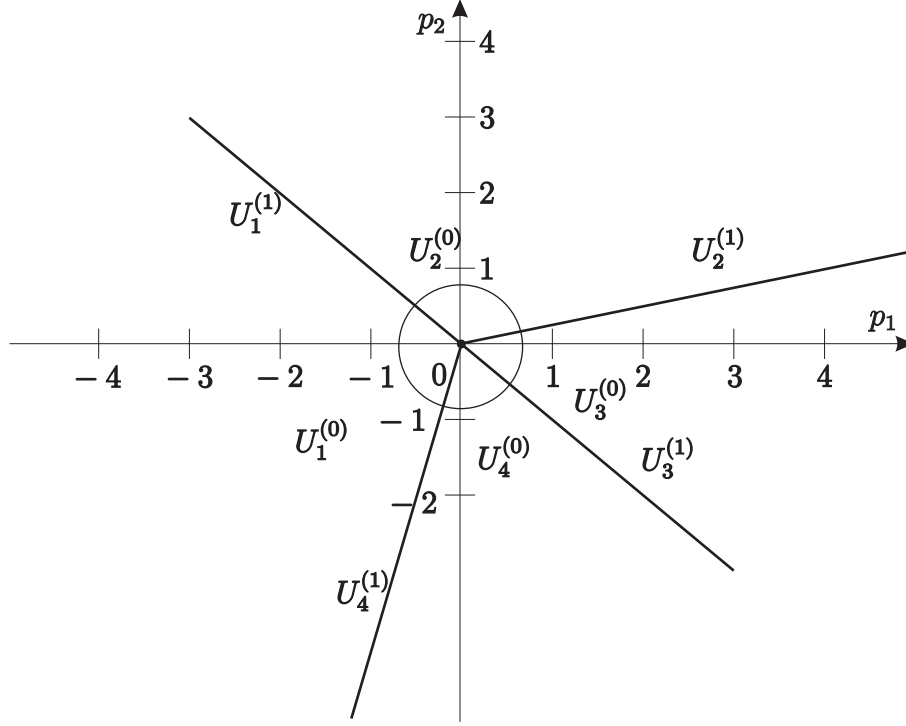


Fig. 2.

The carrier of the equation (1.1) lies in the lattice \mathbb{Z} with the basis $B_1 = Q_1 = (-4, 1)$, $B_2 = Q_3 = (1, 1)$. Expressing the points Q_2, Q_5 through the measuring vectors we get $Q_2 = 4B_2 + B_1$, $Q_5 = B_1 + 2B_2$. Examining the reduced equations that correspond to the bounds $\Gamma_j^{(d)}$ ($d = 0, 1$; $j = 1, 2, 3, 4$) we will study the solutions of the equation (1.1). Note that the reduced equations which correspond to the apexes $\Gamma_2^{(0)}, \Gamma_3^{(0)}, \Gamma_4^{(0)}$

$$\begin{aligned} \hat{f}_2^{(0)} &\stackrel{def}{=} 6w^5 = 0, \\ \hat{f}_3^{(0)} &\stackrel{def}{=} -zw = 0, \\ \hat{f}_4^{(0)} &\stackrel{def}{=} -\alpha = 0 \end{aligned} \quad (2.2)$$

are the algebraic ones and that is why they do not have non-trivial power or non-power solutions.

3 Power expansions, corresponding to the apex $\Gamma_1^{(0)}$ at $\alpha \neq 0$.

The apex $\Gamma_1^{(0)}$ defines the following reduced equation

$$\hat{f}_1^{(0)} \stackrel{def}{=} w_{zzzz} = 0. \quad (3.1)$$

Let us find the reduced solutions

$$w = c_r z^r, \quad c_r \neq 0 \quad (3.2)$$

for $\omega(1, r) \in U_1^{(0)}$. Since $p_1 < 0$ in the cone $U_1^{(0)}$ then $\omega = -1$, $z \rightarrow 0$ and the expansions will be the ascending power series of z . Substituting (3.2) into $\hat{f}_1^{(0)}$ and canceling the result by z^{r-4} we get the characteristic equation

$$\chi(r) \stackrel{def}{=} r(r-1)(r-2)(r-3) = 0 \quad (3.3)$$

with the four roots $r_1 = 0$, $r_2 = 1$, $r_3 = 2$, and $r_4 = 3$. Further we shall examine them separately.

Vector $R = (1, 0)$ that corresponds to the root $r_1 = 0$ being multiplied by ω belongs to the cone $U_1^{(0)}$. Thus we obtain the family $F_1^{(0)}$ of power asymptotics $w = c_0$ with the arbitrary constant $c_0 \neq 0$. The first variation of the equation (3.1)

$$\frac{\delta \hat{f}_1^{(0)}}{\delta w} = \frac{d^4}{dz^4} \quad (3.4)$$

gives the operator

$$L(z) = \frac{d^4}{dz^4} \neq 0 \quad (3.5)$$

with the characteristic polynomial

$$\nu(k) = z^{4-k} L(z) z^k = k(k-1)(k-2)(k-3). \quad (3.6)$$

The equation $\nu(k) = 0$ has four roots $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$. As $\omega = -1$ and $r = 0$ then the cone of the problem is $\mathcal{K} = \{k > 0\}$. It contains $k_2 = 1$, $k_3 = 2$, $k_4 = 3$ which are the critical numbers. Let us investigate the equation obtained from the equation (1.3) after changing of variables $w(z) = c_{r_1} z^{r_1} + y(z)$

$$g(z, y) \stackrel{def}{=} f(z, c_{r_1} z^{r_1} + y). \quad (3.7)$$

Its carrier $S(g)$ contains the points belonging to $S(f)$ and seven other points: $(-2, 2)$, $(-2, 1)$, $(0, 4)$, $(0, 3)$, $(0, 2)$, $(0, 1)$, $(1, 0)$. Displacing all these points on the vector $(-4, 1)$ we get $(2, 1)$, $(2, 0)$, $(4, 3)$, $(4, 2)$, $(4, 1)$, $(4, 0)$, $(5, 1)$, $(4, 4)$, $(0, 0)$, $(4, -1)$, $(5, 0)$. The set S'_+ of the finite sums of the vectors based on these points intersects with the line $q_2 = -1$ at the points

$$\mathbb{K} = \{k = 4 + n, \quad n \in N \cup \{0\}\}. \quad (3.8)$$

The critical numbers do not lie in this set that is why the indexes s in the expression (1.6) belong to the set $\mathbb{K}(1, 2, 3)$.

$$\mathbb{K}(1, 2, 3) = \{k = n, \quad n \in N\}. \quad (3.9)$$

Thus the expansion of the solution corresponding to the reduced equation (3.1) at $r = 0$ is the following

$$w(z) = c_0 + \sum_{k \in \mathbb{K}(1, 2, 3)} c_k z^k \equiv c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \quad (3.10)$$

where all the coefficients are constants, $c_0 \neq 0$, c_1, c_2, c_3 are arbitrary ones and c_k ($k \geq 4$) are uniquely defined. Denote this family as $G_1^{(0)} 1$. Taking into account eight members, the expansion (3.10) can be presented in the form

$$\begin{aligned} w(z) = & c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \left(\frac{1}{24} \alpha - \frac{1}{4} c_0^5 + \frac{5}{12} c_0 c_1^2 + \frac{5}{6} c_0^2 c_2 \right) z^4 + \\ & + \left(\frac{1}{12} c_1^3 + \frac{1}{2} c_0^2 c_3 - \frac{1}{4} c_0^4 c_1 + \frac{1}{120} c_0 + \frac{2}{3} c_0 c_1 c_2 \right) z^5 + \\ & + \left(\frac{2}{9} c_0 c_2^2 + \frac{1}{3} c_0^2 c_4 + \frac{1}{2} c_0 c_1 c_3 + \frac{1}{360} c_1 + \frac{7}{36} c_2 c_1^2 - \frac{1}{12} c_0^4 c_2 - \frac{1}{6} c_0^3 c_1^2 \right) z^6 + \\ & + \left(\frac{5}{21} c_0^2 c_5 + \frac{1}{840} c_2 - \frac{1}{28} c_0^4 c_3 + \frac{8}{21} c_0 c_1 c_4 + \frac{1}{3} c_0 c_2 c_3 + \right. \\ & \left. + \frac{1}{7} c_2^2 c_1 - \frac{1}{7} c_0^3 c_1 c_2 + \frac{13}{84} c_1^2 c_3 - \frac{1}{14} c_0^2 c_1^3 \right) z^7 + \dots \end{aligned}$$

The cone of the problem is $\mathcal{K} = \{k > 1\}$ in the case $r_2 = 1$. Consequently there are two critical numbers $k_2 = 2$, $k_3 = 3$. Likewise the previous case we find the solution expansion

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \quad (3.11)$$

that is generated by the power asymptotic $\mathcal{F}_1^{(0)} 2$: $w = c_1 z$. Here $c_1 \neq 0$, c_2 and c_3 are arbitrary constants.

For the root $r_2 = 2$ the cone of the problem is $\mathcal{K} = \{k > 2\}$. It contains $k_3 = 3$ that is the unique critical number. The power expansion corresponding to the asymptotic $\mathcal{F}_1^{(0)} 3$: $w = c_2 z^2$ takes the form

$$w(z) = c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k. \quad (3.12)$$

Again $c_2 \neq 0$, c_3 are arbitrary constants. Denote this family as $G_1^{(0)} 3$.

For the root $r_3 = 3$ the cone of the problem is $\mathcal{K} = \{k > 3\}$. There are no critical numbers in this case. The expansion of the solution $G_1^{(0)} 4$ corresponding

to the power asymptotic $\mathcal{F}_1^{(1)}4$: $w = c_3 z^3$ can be written as

$$w(z) = c_3 z^3 + \sum_{k=4}^{\infty} k z^k. \quad (3.13)$$

Obtained expansions converge for small $|z|$. The existence and the analyticity of the expansions (3.10), (3.11), (3.12), (3.13) follow from the Cauchy theorem. The apex does not define non-power asymptotics and exponential additions.

4 Power expansions, corresponding to the edge $\Gamma_1^{(1)}$ at $\alpha \neq 0$.

The edge $\Gamma_1^{(1)}$ is characterized by the reduced equation

$$\hat{f}_1^{(1)}(z, w) \stackrel{\text{def}}{=} w_{zzzz} - 10 w^2 w_{zz} - 10 w w_z^2 + 6 w^5 = 0 \quad (4.1)$$

and the normal cone $U_1^{(1)} = \{-\lambda(1, -1), \lambda > 0\}$. Therefore $\omega = -1$, i.e. $z \rightarrow 0$ and $r = -1$. Consequently the solution of the equation (4.1) should be looked for in the form $w = c_{-1} z^{-1}$. From the determining equation

$$c_{-1}(3 c_{-1}^4 - 15 c_{-1}^2 + 12) = 0 \quad (4.2)$$

we find the values of coefficient c_{-1} (not equal to zero): $c_{-1}^{(1)} = 1$, $c_{-1}^{(2)} = -1$, $c_{-1}^{(3)} = 2$, $c_{-1}^{(4)} = -2$. Hence we have four families of power asymptotics

$$\mathcal{F}_1^{(1)}1 : w = z^{-1}; \quad \mathcal{F}_1^{(1)}2 : w = -z^{-1}; \quad (4.3)$$

$$\mathcal{F}_1^{(1)}3 : w = 2z^{-1}; \quad \mathcal{F}_1^{(1)}4 : w = -2z^{-1}. \quad (4.4)$$

Let us compute the corresponding critical numbers. The first variation

$$\frac{\delta f_2^{(1)}}{\delta w} = \frac{d^4}{dz^4} - 20 w_{zz} w - 10 w^2 \frac{d^2}{dz^2} - 10 w_z^2 - 10 w w_z \frac{d}{dz} + 30 w^4 \quad (4.5)$$

applied to the the solutions (4.4) yields the operator

$$\mathcal{L}^{(1,2)}(z) = \frac{d^4}{dz^4} - \frac{10}{z^2} \frac{d^2}{dz^2} + \frac{20}{z^3} \frac{d}{dz} - \frac{20}{z^4}. \quad (4.6)$$

Its characteristic polynomial is

$$\nu(k) = k^4 - 6 k^3 + k^2 + 24 k - 20. \quad (4.7)$$

The equation $\nu(k) = 0$ has the roots $k_1 = -2$, $k_2 = 1$, $k_3 = 2$, $k_4 = 5$.

With reference to the solutions (4.5) variation (4.6) gives the operator

$$\mathcal{L}^{(3,4)}(z) = \frac{d^4}{dz^4} - \frac{40}{z^2} \frac{d^2}{dz^2} + \frac{80}{z^3} \frac{d}{dz} - \frac{280}{z^4}, \quad (4.8)$$

Its characteristic polynomial

$$\nu(k) = k^4 - 6k^3 - 29k^2 + 114k + 280, \quad (4.9)$$

has the roots $k_1 = -4$, $k_2 = -2$, $k_3 = 5$, $k_4 = 7$. The cone of the problem looks like

$$\mathcal{K} = \{k > -2\}. \quad (4.10)$$

Thus for the power asymptotics (4.4) there are three critical numbers (three roots of the characteristic polynomial belong to the cone of the problem) and for the power asymptotics (4.5) there are only two critical numbers. The shifted carrier of the power asymptotics (4.4), (4.5) is the vector $(-1, -1) = -B_2$. It belongs to the lattice that consists of the points $Q = m(-4, 1) + l(1, 1)$ where m and l are the whole numbers. These points intersect with the line $q_2 = -1$ if $m + l = -1$, i.e. $l = -m - 1$. As the cone of the problem is (4.11) then

$$\mathbf{K} = \{k = -1 + 5m, \ m \in \mathbb{N}\}. \quad (4.11)$$

Now we will look for the solution expansions generated by the families (4.4). The sets $\mathbf{K}(1)$, $\mathbf{K}(1, 2)$ and $\mathbf{K}(1, 2, 5)$ can be written as

$$\begin{aligned} \mathbf{K}(1) &= \{k = -1 + 5m + 2l, \ l, m \in \mathbb{N} \cup \{0\}, \ l + m \neq 0\} = \\ &= \{1, 3, 4, 5, 6, 7, 8, \dots\}; \end{aligned} \quad (4.12)$$

$$\begin{aligned} \mathbf{K}(1, 2) &= \{k = -1 + 5m + 2l + 3k, \ l, m, k \in \mathbb{N} \cup \{0\}, \ m + l + k \neq 0\} = \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, \dots\}; \end{aligned} \quad (4.13)$$

$$\begin{aligned} \mathbf{K}(1, 2, 5) &= \{k = -1 + 5m + 2l + 3k + 6n, \ n, m, k, l \in \mathbb{N} \cup \{0\}, \ m + n + k + l \neq 0\} = \\ &= \{n, \ n \in \mathbb{N}\}. \end{aligned} \quad (4.14)$$

In this case the solution expansions are

$$w(z) = \frac{c_{-1}^{(1,2)}}{z} + \sum_{k \in \mathbf{K}(1,2,5)} c_k^{(1,2)} z^k. \quad (4.15)$$

Denote these families as $G_1^{(1)}1$ and $G_1^{(1)}2$. Evidently the critical number 1 does not belong to the set \mathbf{K} that is why the compatibility condition for c_1 holds automatically and in consequence c_1 is an arbitrary constant. The critical number 2 also does not belong to the sets \mathbf{K} , $\mathbf{K}(1)$ and thus c_2 is an arbitrary

constant. However the critical number 5 lies in the sets $\mathbf{K}(1)$ and $\mathbf{K}(1, 2)$. As a result it is necessary to verify that the compatibility condition for c_5 is true. It turns out that it is so then c_5 is an arbitrary constant. The three-parametric power expansion that corresponds to the power asymptotics (4.4) (at $c_{-1} = 1$) is the following

$$\begin{aligned} w(z) = & \frac{1}{z} + c_1 z + c_2 z^2 + \frac{5}{2} c_1^2 z^3 + \left(\frac{5}{3} c_1 c_2 - \frac{1}{36} \alpha - \frac{1}{36} \right) z^4 + c_5 z^5 + \\ & + \left(\frac{5}{8} c_2 c_3 + \frac{37}{48} c_1^2 c_2 - \frac{7}{288} c_1 \alpha - \frac{13}{720} c_1 \right) z^6 + \\ & + \left(\frac{17}{27} c_1 c_2^2 - \frac{17}{1620} c_2 - \frac{1}{18} c_1^4 - \frac{1}{81} c_2 \alpha - \frac{5}{54} c_1^2 c_3 + \frac{5}{9} c_1 c_5 + \frac{11}{54} c_3^2 \right) z^7 + \dots \end{aligned}$$

Here $c_i \equiv c_i^{(1)}$, $i = 1, 2, 5$.

The carrier of the power expansions generated by the families (4.5) is defined by the sets

$$\begin{aligned} \mathbf{K}(5) = \{k = -1 + 5m + 6l, l, m \in \mathbb{N} \cup \{0\}, m + l \neq 0\} = \\ = \{4, 5, 9, 10, 11, 14, 15, 16, 17, 19, \dots\}; \end{aligned} \quad (4.16)$$

$$\begin{aligned} \mathbf{K}(5, 7) = \{k = -1 + 5m + 6l + 8k, m, l, k \in \mathbb{N} \cup \{0\}, m + l + k \neq 0\} = \\ = \{4, 5, 7, 9, 10, 11, 12, 13, 14, \dots\}. \end{aligned} \quad (4.17)$$

The solution expansions in this case are

$$w(z) = \frac{c_{-1}^{(3,4)}}{z} + \sum_{k \in \mathbf{K}(5,7)} c_k^{(3,4)} z^k. \quad (4.18)$$

Denote these families as $G_1^{(1)}3, G_1^{(1)}4$. Obviously the critical numbers 5 and 7 do not lie in the set \mathbf{K} and besides the critical number 7 does not belong to the set $\mathbf{K}(5)$. For the values 5, 7 the compatibility condition holds automatically and as a consequence the coefficients c_5, c_7 are arbitrary ones. The two-parametric power expansion that corresponds to the reduced solutions (4.5) (at $c_{-1} = 2$) is

$$\begin{aligned} w(z) = & \frac{2}{z} + 2 z^{-1} + \left(\frac{1}{72} + \frac{1}{144} \alpha \right) z^4 + c_5 z^5 + c_7 z^7 + \\ & + \left(\frac{17}{370656} + \frac{59}{1482624} \alpha + \frac{25}{2965248} \alpha^2 \right) z^9 + c_5 \left(\frac{19}{3780} + \frac{1}{432} \alpha \right) z^{10} + \\ & + \frac{11}{78} c_5^2 z^{11} + c_7 \left(\frac{127}{35280} + \frac{1}{576} \alpha \right) z^{12} + \dots, \end{aligned}$$

where $c_i \equiv c_i^{(3)}$, $i = 5, 7$. Obtained expansions converge for small $|z|$. The exponential additions for the expansions (4.15), (4.18) do not exist. The reduced equation (4.1) does not have non-power solutions.

5 Power expansions, corresponding to the edge $\Gamma_2^{(1)}$ at $\alpha \neq 0$.

The edge $\Gamma_2^{(1)}$ is characterized by the reduced equation

$$\hat{f}_2^{(1)}(z, w) \stackrel{\text{def}}{=} 6w^5 - zw = 0 \quad (5.1)$$

and the normal cone $U_2^{(1)} = \{\lambda(4, 1) = 4\lambda(1, 1/4), \lambda > 0\}$. It means that $r = 1/4$, $\omega = 1$, $z \rightarrow \infty$ and the solution of the reduced equation is $y = c_{1/4}z^{1/4}$. Substitution this expression into (5.1) and cancelation the result by $z^{5/4}$ yields the determining equation $c_{1/4}(6c_{1/4}^4 - 1) = 0$. Thus we have four families of power asymptotics

$$\mathcal{F}_2^{(1)}1: \quad w = c_{1/4}^{(1)} z^{1/4}, \quad c_{1/4}^{(1)} = \left(\frac{1}{6}\right)^{1/4}; \quad (5.2)$$

$$\mathcal{F}_2^{(1)}2: \quad w = c_{1/4}^{(2)} z^{1/4}, \quad c_{1/4}^{(2)} = -\left(\frac{1}{6}\right)^{1/4}; \quad (5.3)$$

$$\mathcal{F}_2^{(1)}3: \quad w = c_{1/4}^{(3)} z^{1/4}, \quad c_{1/4}^{(3)} = i\left(\frac{1}{6}\right)^{1/4}; \quad (5.4)$$

$$\mathcal{F}_2^{(1)}4: \quad w = c_{1/4}^{(4)} z^{1/4}, \quad c_{1/4}^{(4)} = -i\left(\frac{1}{6}\right)^{1/4}. \quad (5.5)$$

Since the reduced equation (5.1) is algebraic and the roots of the determining equation are simple then asymptotics do not have proper (and consequently critical) numbers and $\nu(k) \equiv \text{const} \neq 0$. The shifted carrier of the power asymptotics (5.2) – (5.5) gives a vector $B = (1/4, -1)$. Points belonging to the lattice generated by vectors B_2, B are the following $Q = (q_1, q_2) = m(1, 1) + l(1/4, -1) = (m + l/4, m - l)$ where m, l are whole numbers. At the line $q_2 = -1$ we have $q_2 = -1$ and $q_1 = -1 + 5/4l$. Taking into consideration that the cone of the problem is $\mathcal{K} = \{k < 1/4\}$ we find the set \mathbf{K}

$$\mathbf{K} = \{k = -1 - 5l/4, \quad l \in \mathbb{N} \cup \{0\}\}, \quad (5.6)$$

The expansions of the solutions can be written as

$$G_2^{(1)}n: \quad w(z) = \varphi^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \sum_{l=0}^{\infty} c_{-1-5l/4}^{(n)} z^{-1-5l/4}. \quad (5.7)$$

In this expression coefficients $c_{-1-5l/4}^{(n)}$ can be sequentially computed. The calculation of the coefficients c_{-1} yields $c_{-1} = \alpha/4$. Taking into account four

terms, the expansions are

$$w(z) = \varphi^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \frac{1}{4} \frac{\alpha}{z} - \frac{5}{16} \frac{(c_{1/4}^{(n)})^3 (1 + 3\alpha^2)}{z^{9/4}} + \\ + \frac{5}{128} \frac{(c_{1/4}^{(n)})^2 \alpha (29 + 24\alpha^2)}{z^{7/2}} + \dots$$

It is likely that the obtained expansions diverge. Non-power asymptotics do not correspond to the edge $\Gamma_2^{(1)}$ but it generates exponential additions which will be computed later.

6 Power expansions, corresponding to the edge $\Gamma_3^{(1)}$ at $\alpha \neq 0$.

The edge $\Gamma_3^{(1)}$ is characterized by the reduced equation

$$\hat{f}_3^{(1)}(z, w) \stackrel{def}{=} -zw - \alpha = 0 \quad (6.1)$$

and the normal cone $U_3^{(1)} = \{\lambda(1, -1), \lambda > 0\}$. In this case $r = -1$, $\omega = 1$, $z \rightarrow \infty$ and power asymptotic can be presented in the form

$$\mathcal{F}_3^{(1)} 1 : w = \frac{c_{-1}}{z}, \quad c_{-1} = -\alpha. \quad (6.2)$$

As the equation (6.1) is algebraic, then its solutions do not have critical numbers and

$$\nu(k) = z^{-1} \frac{\delta f_3^{(1)}}{\delta y} = -1. \quad (6.3)$$

The cone of the problem is $\mathcal{K} = \{k < -1\}$. The shifted carrier of the power asymptotic (6.2) gives the vector $(-1, -1) = -B_2$, which belongs to the lattice generated by the carrier of the studied equation. That is why

$$\mathbf{K} = \{k = -1 - 5m, \quad m \in \mathbb{N}\}. \quad (6.4)$$

So we have determined the power expansion corresponding to the asymptotic (6.2)

$$G_3^{(1)} : w(z) = \psi(z) = \frac{c_{-1}}{z} + \sum_{m=1}^{\infty} c_{-1-5m} z^{-1-5m}. \quad (6.5)$$

In this expression all coefficients can be sequentially found. Again taking into account four terms, it can be rewritten as

$$w(z) = \psi(z) = -\frac{\alpha}{z} - 6 \frac{c}{z^6} - 12 \frac{c(15\alpha^4 - 295\alpha^2 + 1512)}{z^{11}} - \\ - 216 \frac{c(35\alpha^8 - 2090\alpha^6 + 49755\alpha^4 - 528180\alpha^2 + 2018016)}{z^{16}} + \dots,$$

where $c = \alpha(\alpha-1)(\alpha-2)(\alpha+1)(\alpha+2)$ and all other not written out coefficients are proportional to this factor. Unless $c = 0$ the expansion (6.5) seems to be divergent one for all $|z|^{-1} \neq 0$.

The reduced equation (6.1) does not have non-power solutions. The edge $\Gamma_3^{(1)}$ defines exponential additions which will be found below.

7 Power expansions, corresponding to the edge $\Gamma_4^{(1)}$ at $\alpha \neq 0$.

The edge $\Gamma_4^{(1)}$ defines the following reduced equation

$$\hat{f}_4^{(1)}(z, w) \stackrel{\text{def}}{=} w_{zzzz} - \alpha = 0 \quad (7.1)$$

and the normal cone $U_4^{(1)} = \{-\lambda(1, 4), \lambda > 0\}$. Thus $\omega = -1$, i.e. $z \rightarrow 0$ $r = 4$ and we have the unique family of power asymptotics

$$\mathcal{F}_4^{(1)} 1 : w = c_4 z^4, \quad c_4 = \frac{\alpha}{24}. \quad (7.2)$$

Let us find the critical numbers. The first variation of the equation (7.1) is

$$\frac{\delta \hat{f}_4^{(1)}}{\delta w} = \frac{d^4}{dz^4}. \quad (7.3)$$

The proper numbers are $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$. None of them belongs to the cone of the problem $\mathcal{K} = \{k > 4\}$. Consequently in this case there are no critical numbers. The shifted carrier of (7.2) is equal to the vector B_1 and the lattice generated by the vectors B_1, B_2 corresponds to the power asymptotic (7.2). Because of this the set \mathbf{K} is

$$\mathbf{K} = \{k = 4 + 5m, \quad m \in \mathbb{N}\}. \quad (7.4)$$

The power expansion can be written as

$$w(z) = z^4 \left(\frac{\alpha}{24} + \sum_{m=1}^{\infty} c_{4+5m} z^{5m} \right). \quad (7.5)$$

The coefficient c_{4+5m} can be uniquely computed. The first three terms of the found expansion $G_4^{(1)} 1$ are

$$w(z) = \alpha \left(\frac{1}{24} z^4 + \frac{1}{72576} z^9 + \left(\frac{1}{1743565824} + \frac{5}{5930496} \alpha^2 \right) z^{14} + \dots \right).$$

The expansion (7.5) can be regarded as the special case of the expansion (3.10) at $c_0 = c_1 = c_2 = c_3 = 0$. It converge for small $|z|$. The existence,

the uniqueness and the analyticity of such expansion follow from the above mentioned Cauchy theorem.

The edge $\Gamma_4^{(1)}$ does not define exponential additions and non-power asymptotics.

8 General properties of the equation (1.3) at $\alpha = 0$.

If $\alpha = 0$ the fourth-order analogue to the second Painlevé equation is

$$\tilde{f}(z, w) \stackrel{\text{def}}{=} w_{zzzz} - 10w^2w_{zz} - 10ww_z^2 + 6w^5 - zw = 0. \quad (8.1)$$

The carrier $S(\tilde{f})$ of this equation consists of the four points: $Q_1 = (-4, 1)$, $Q_2 = (0, 5)$, $Q_3 = (1, 1)$, $Q_4 = (-2, 3)$. Their convex hull is the triangle with the apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3$) and edges $\Gamma_1^{(1)} = \{Q_1, Q_2\}$, $\Gamma_2^{(1)} = \{Q_2, Q_3\}$, $\Gamma_3^{(1)} = \{Q_3, Q_1\}$ (fig. 3).

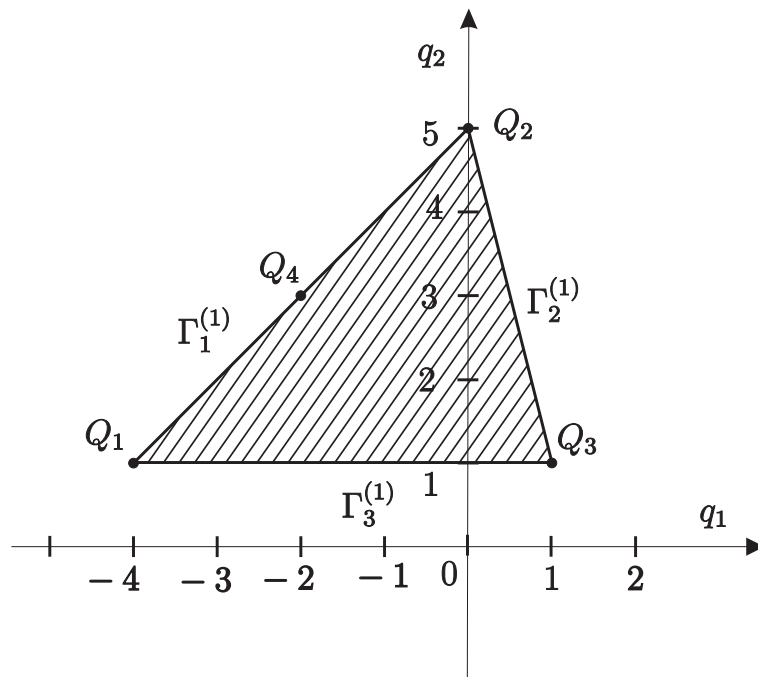


Fig. 3.

The external normals N_j to the edges $\Gamma_j^{(1)}$ are the vectors $N_1 = (-1, 1)$, $N_2 = (4, 1)$, $N_3 = (0, -1)$. Normal cones $U_j^{(1)}$ of the edges $\Gamma_j^{(1)}$ can be written as

$$U_j^{(1)} = \{P = \lambda N_j, \lambda > 0\}, \quad j = 1, 2, 3. \quad (8.2)$$

Normal cones $U_j^{(0)}$ of the apexes $\Gamma_j^{(0)}$ and the normal cones (8.2) are presented at the fig. 4. The shifted carrier of the equation (8.1) lies in the lattice with the basis $B_3 = (-3, 2)$ and $B_4 = (-1, 4)$.

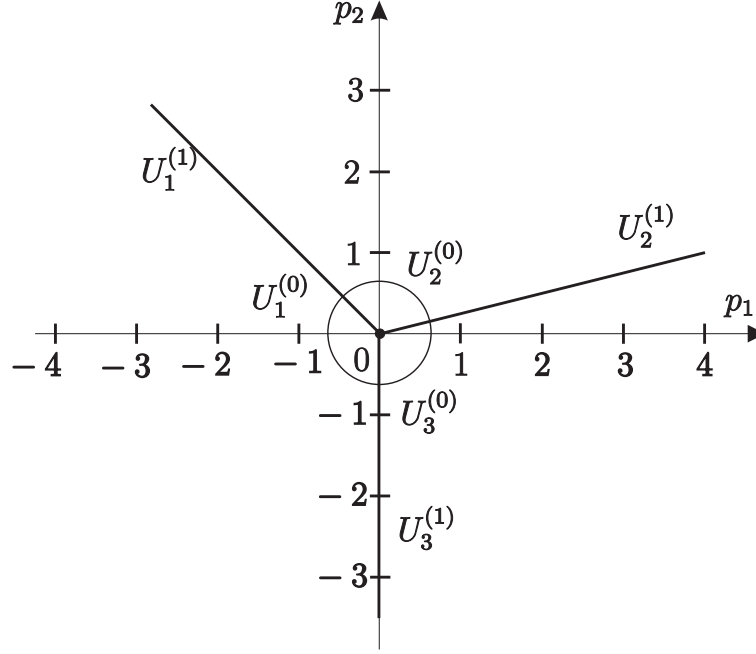


Fig. 4.

Further we will study the reduced equations corresponding to the apexes $\Gamma_j^{(0)}$ and to the edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3$). Again the reduced equations

$$\hat{f}_2^{(0)}(z, w) \stackrel{def}{=} 6w^5 = 0, \quad \hat{f}_3^{(0)}(z, w) \stackrel{def}{=} -zw = 0, \quad (8.3)$$

which are defined by the apexes $\Gamma_2^{(0)}$, $\Gamma_3^{(0)}$ accordingly are the algebraic ones. Because of this they do not have non-trivial power or non-power solutions.

9 Power expansions, corresponding to the apex $\Gamma_1^{(0)}$ and to the edge $\Gamma_1^{(1)}$ at $\alpha = 0$.

The apex $\Gamma_1^{(0)}$ is characterized by the following reduced equation

$$\hat{f}_1^{(0)}(z, w) \stackrel{def}{=} w_{zzzz} = 0. \quad (9.1)$$

This case is similar to the one discussed in the section 3. Consequently there are four families of power expansions: four-parametric family $\tilde{G}_1^{(0)}1$, tree-parametric family $\tilde{G}_1^{(0)}2$, two-parametric family $\tilde{G}_1^{(0)}3$ and one-parametric family $\tilde{G}_1^{(0)}4$.

Examining by analogy with the section 4 the reduced equation which corresponds to the edge $\Gamma_1^{(1)}$

$$\hat{f}_1^{(1)}(z, w) \stackrel{def}{=} w_{zzzz} - 10 w^2 w_{zz} - 10 w w_z^2 + 6 w^5 = 0, \quad (9.2)$$

we get two tree-parametric families $\tilde{G}_1^{(1)}1$, $\tilde{G}_1^{(1)}2$ and two two-parametric families $\tilde{G}_1^{(1)}3$, $\tilde{G}_1^{(1)}4$ of power expansions.

The wave in these designations means that in the similar expressions for the equation (1.3) it is provided that $\alpha = 0$.

10 Power expansions, corresponding to the edge $\Gamma_2^{(1)}$ at $\alpha = 0$.

The edge $\Gamma_2^{(1)}$ defines the reduced equation which can be written as

$$\hat{f}_2^{(1)}(z, w) \stackrel{def}{=} 6 w^5 - zw = 0. \quad (10.1)$$

Following the analysis of the section 5 we get that $r = 1/4$, $\omega = 1$, $z \rightarrow \infty$ and that there are four families of power asymptotics: $\tilde{\mathcal{F}}_2^{(1)}1$ (5.2), $\tilde{\mathcal{F}}_2^{(1)}2$ (5.3), $\tilde{\mathcal{F}}_2^{(1)}3$ (5.4), $\tilde{\mathcal{F}}_2^{(1)}4$ (5.5). The shifted carrier of the reduced solutions is the vector $(1/4, -1)$. Together with the vectors B_3 , B_4 it generates the lattice with the basis $(1/4, -1)$ and $(-3, 2)$. The points of this lattice are $Q = (q_1, q_2) = m(1/4, -1) + l(-3, 2) = (m/4 - 3l, -m + 2l)$. At the line $q_2 = -1$ we have $q_1 = 1/4 + l/2 - 3l = 1/4 - 5l/2$. As the cone of the problem is $\mathcal{K} = \{k < 1/4\}$, then the set

$$\mathbf{K} = \{k = 1/4 - 5l/2, \quad l \in \mathbb{N}\} \quad (10.2)$$

differs from the similar in the section 5. Hence the power expansions can be presented in the form

$$\tilde{G}_2^{(1)}n : \quad w(z) = \varphi^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \sum_{l=1}^{\infty} c_{1/4-5l/2}^{(n)} z^{1/4-5l/2}. \quad (10.3)$$

All coefficients in the expression (10.3) can be sequentially computed. Taking into account three terms these expansions can be rewritten as

$$w(z) = \varphi^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} - \frac{5}{16} \frac{(c_{1/4}^{(n)})^3}{z^{9/4}} - \frac{297}{512} \frac{c_{1/4}^{(n)}}{z^{19/4}} + \dots$$

It is likely that obtained expressions are divergent ones.

11 Non-power expansions, corresponding to the edge $\Gamma_3^{(1)}$ in the case $\alpha = 0$.

The reduced equation which corresponds to the edge $\Gamma_3^{(1)}$

$$\hat{f}_2^{(3)}(z, w) \stackrel{\text{def}}{=} w_{zzzz} - zw = 0 \quad (11.1)$$

does not possess solutions in the form $w = c_r z^r$, $c_r \neq 0$ (it has a trivial solution $w \equiv 0$). The edge $\Gamma_3^{(1)}$ defines non-power asymptotics of the equation (8.1). It is horizontal, consequently in order to find the solutions of the reduced equation (11.1) it is necessary to make the logarithmic transformation

$$y = \frac{d \ln w}{dz}. \quad (11.2)$$

Hence derivatives of $w(z)$ are

$$\begin{aligned} w' &= yw, \quad w'' = (y' + y^2)w, \quad w''' = (y'' + 3yy' + y^3)w, \\ w'''' &= (y''' + 4yy'' + 3(y')^2 + 6y^2y' + y^4)w. \end{aligned}$$

After application of this transformation and after cancelation of the result by w the equation (11.1) can be rewritten as

$$h(z, y) \stackrel{\text{def}}{=} y''' + 4yy'' + 3(y')^2 + 6y^2y' + y^4 - z = 0. \quad (11.3)$$

The carrier $S(h)$ of this equation consists of five points: $M_1 = (-3, 1)$, $M_2 = (0, 4)$, $M_3 = (1, 0)$, $M_4 = (-2, 2)$, $M_5 = (-1, 3)$. Their convex hull $\Gamma(h)$ is the triangle with the apexes $\tilde{\Gamma}_j^{(0)} = M_j$, $j = 1, 2, 3$ and the edges $\tilde{\Gamma}_1^{(1)} = \{M_1, M_2\}$, $\tilde{\Gamma}_2^{(1)} = \{M_2, M_3\}$, $\tilde{\Gamma}_3^{(1)} = \{M_3, M_1\}$ (fig. 5).

The external normals \tilde{N}_j to the edges $\tilde{\Gamma}_j^{(1)}$ are $\tilde{N}_1 = (-1, 1)$, $\tilde{N}_2 = (4, 1)$, $\tilde{N}_3 = (-1, -4)$. The normal cones of the apexes and of the edges are presented at the fig. 6. The shifted carrier of the equation (11.3) lies in the lattice with the basis $B_5 = (1, 1)$, $B_6 = (1, -4)$. In this case the cone of the problem is $p_1 + p_2 > 0$. It intersects with the normal cones $\tilde{U}_2^{(0)}$, $\tilde{U}_3^{(0)}$, $\tilde{U}_2^{(1)}$. That is why later it is sufficient to study the reduced equations which correspond to the bounds $\tilde{\Gamma}_2^{(0)}$, $\tilde{\Gamma}_3^{(0)}$, $\tilde{\Gamma}_2^{(1)}$ only. The first two of them are characterized by the algebraic reduced equations $y^4 = 0$, $-z = 0$, accordingly. They do not give suitable solutions. The edge $\tilde{\Gamma}_2^{(1)}$ defines algebraic reduced equation

$$y^4 - z = 0, \quad (11.4)$$

which has four suitable solutions $y^{(n)}(z) = c_{1/4}^{(n)} z^{1/4}$ where $c_{1/4}^{(n=\{1,2\})} = \pm 1$, $c_{1/4}^{(n=\{3,4\})} = \pm i$ and $r = 1/4$, $\omega = 1$. These solutions do not have critical numbers. The shifted carrier of the equation (11.3) is the vector $(1/4, -1) = 1/4 B_6$. Thus

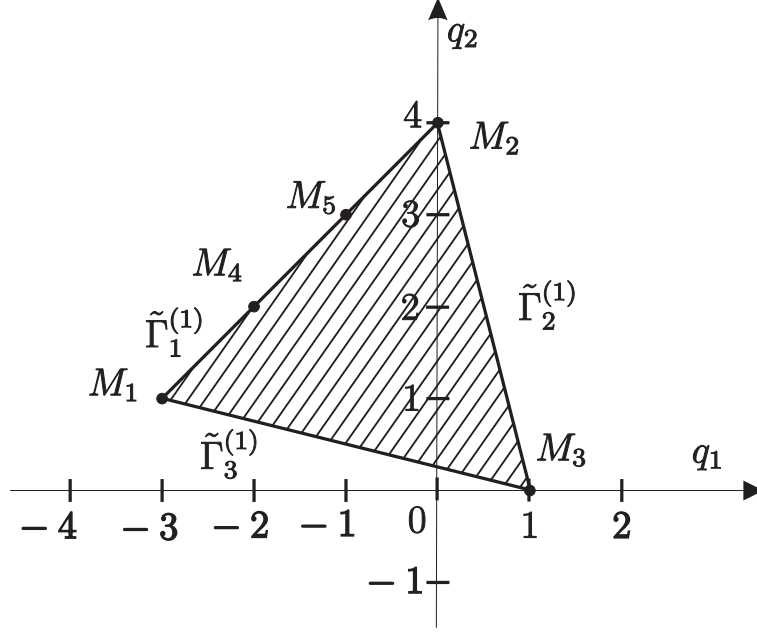


Fig. 5.

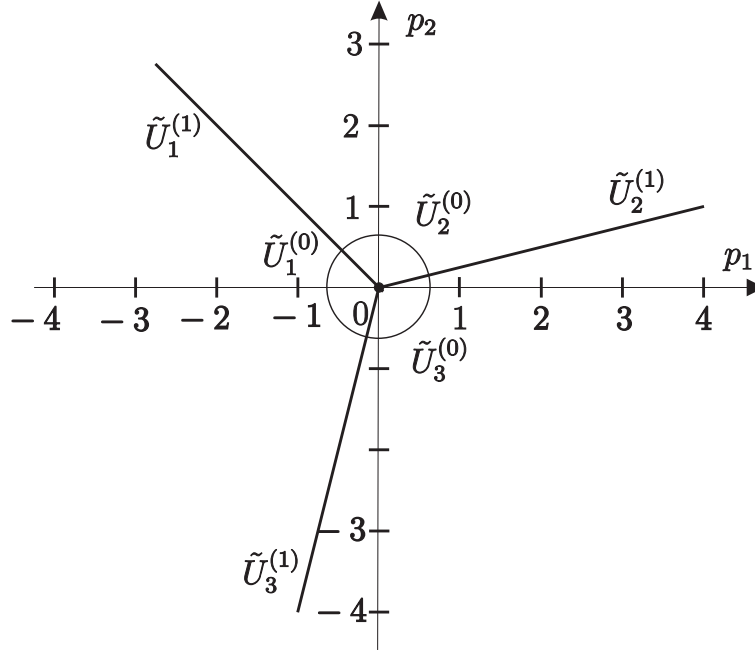


Fig. 6.

we have a lattice with the basis $(1/4, -1)$, $(1, 1)$. Taking into account the fact that the cone of the problem is $\mathcal{K} = \{k < 1/4\}$, we find the set \mathbf{K}

$$\mathbf{K} = \{k = 1/4 - 5l/4, \quad l \in \mathbb{N}\}. \quad (11.5)$$

The equation (11.3) has solutions in the form of the series

$$y^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \sum_{l=1}^{\infty} c_{1/4-5l/4}^{(n)} z^{1/4-5l/4}, \quad n = 1, 2, 3, 4, \quad (11.6)$$

where the coefficients $c_{1/4-5l/4}^{(n)}$ are uniquely defined. The coefficient $c_{-1}^{(n)}$ does not depend on n and $c_{-1}^{(n)} = -3/8$. Introducing the new variable $\eta = \ln w$ we get

$$\eta = \int y dz = \frac{4}{5} c_{1/4}^{(n)} z^{5/4} + C_0 - \frac{3}{8} \ln z + \sum_{l=2}^{\infty} \frac{4c_{1/4-5l/4}^{(n)}}{5(1-l)} z^{5(1-l)/4}. \quad (11.7)$$

In this expression C_0 is a constant of integration. Returning to the variable $w(z)$ yields

$$w(z) = \frac{C_1}{z^{3/8}} \exp \left[\frac{4}{5} c_{1/4}^{(n)} z^{5/4} + \sum_{l=2}^{\infty} \frac{4c_{1/4-5l/4}^{(n)}}{5(1-l)} z^{5(1-l)/4} \right]. \quad (11.8)$$

Taking into account two terms of the series the obtained non-power asymptotics of the equation (8.1) can be written as

$$w(z) = \frac{C_1}{z^{3/8}} \exp \left[\frac{4}{5} c_{1/4}^{(n)} z^{5/4} + \frac{9}{32} \frac{1}{c_{1/4}^{(n)} z^{5/4}} + \frac{45}{256} \frac{1}{(c_{1/4}^{(n)})^2 z^{5/2}} + \dots \right].$$

Providing that $w \rightarrow 0$ at $z \rightarrow \infty$, we should take the number n in (11.8) so that

$$Re(c_{1/4}^{(n)} z^{5/4}) < 0. \quad (11.9)$$

Consequently we have found four one-parametric non-power asymptotics for the solutions of the equation (8.1): $\tilde{G}_3^{(1)} 1$, $\tilde{G}_3^{(1)} 2$, $\tilde{G}_3^{(1)} 3$, $\tilde{G}_3^{(1)} 4$.

12 Exponential additions to the solutions.

Let us find exponential additions to the obtained solutions. Since the equations $\partial \hat{f}_j^{(d)} / \partial w^{\Pi(f)} = 0$ do not have solutions for $\forall (d = 0, 1; j = 1, 2, 3, 4)$ and the condition $\Pi(\hat{f}_j^{(d)}) < \Pi(f) = 4$ holds for the edges $\Gamma_2^{(1)}$ and $\Gamma_3^{(1)}$ only (in the case $\alpha \neq 0$) then these very edges define exponential additions which are to be computed. At $\alpha = 0$ the condition $\Pi(\hat{\tilde{f}}_j^{(d)}) < \Pi(\tilde{f}) = 4$ is true for the edge $\Gamma_2^{(1)}$, consequently solutions corresponding to this edge possess exponential additions which can be calculated as it will be done for the edge $\Gamma_2^{(1)}$ of the equation (1.3).

13 Exponential additions of the first level, corresponding to the edge $\Gamma_2^{(1)}$ at $\alpha \neq 0$.

To begin with we will look for the exponential additions of the first level $u^{(n)}(z)$ to the expansions (5.7), i.e. we will look for the functions

$$w(z) = \varphi^{(n)}(z) + u^{(n)}(z), \quad n = 1, 2, 3, 4. \quad (13.1)$$

The reduced equation for the addition $u^{(n)}(z)$ is a linear equation

$$M_n^{(1)}(z)u^{(n)}(z) = 0, \quad (13.2)$$

where $M_n^{(1)}(z)$ is the first variation at the solutions $w(z) = \varphi^{(n)}(z)$. As long as

$$\frac{\delta f}{\delta w} = \frac{d^4}{dz^4} - 10w^2 \frac{d^2}{dz^2} - 20ww_z \frac{d}{dz} - 20w_{zz}w - 10w_z^2 + 30w^4 - z, \quad (13.3)$$

then

$$M_n^{(1)}(z) = \frac{d^4}{dz^4} - 10(\varphi^{(n)})^2 \frac{d^2}{dz^2} - 20\varphi^{(n)}\varphi_z^{(n)} \frac{d}{dz} - 20\varphi_{zz}^{(n)}\varphi^{(n)} - 10(\varphi_z^{(n)})^2 + 30(\varphi^{(n)})^4 - z. \quad (13.4)$$

And equation (13.2) can be rewritten as

$$\begin{aligned} & \frac{d^4 u^{(n)}}{dz^4} - 10(\varphi^{(n)})^2 \frac{d^2 u^{(n)}}{dz^2} - 20\varphi^{(n)}\varphi_z^{(n)} \frac{du^{(n)}}{dz} - \\ & - [10(2\varphi_{zz}^{(n)}\varphi^{(n)} + (\varphi_z^{(n)})^2 - 3(\varphi^{(n)})^4) + z] u^{(n)} = 0, \quad n = 1, 2, 3, 4. \end{aligned} \quad (13.5)$$

Let us make the change of the variables

$$\zeta^{(n)}(z) = \frac{d \ln u^{(n)}(z)}{dz}. \quad (13.6)$$

Computing the derivatives of the function $u^{(n)}(z)$ yields

$$\begin{aligned} \frac{du^{(n)}}{dz} &= \zeta^{(n)}u^{(n)}, \quad \frac{d^2 u^{(n)}}{dz^2} = \zeta_z^{(n)}u^{(n)} + (\zeta^{(n)})^2 u^{(n)}, \\ \frac{d^3 u^{(n)}}{dz^3} &= \zeta_{zz}^{(n)}u^{(n)} + 3\zeta^{(n)}\zeta_z^{(n)}u^{(n)} + (\zeta^{(n)})^3 u^{(n)}, \\ \frac{d^4 u^{(n)}}{dz^4} &= \zeta_{zzz}^{(n)}u^{(n)} + 4\zeta^{(n)}\zeta_{zz}^{(n)}u^{(n)} + 3(\zeta_z^{(n)})^2 u^{(n)} + 6(\zeta^{(n)})^2 \zeta_z^{(n)}u^{(n)} + (\zeta^{(n)})^4 u^{(n)}. \end{aligned}$$

Substituting these expressions into the equation (13.5) and setting to zero the factor at $u^{(n)}(z)$ we get the equation

$$\begin{aligned} & \zeta_{zzz}^{(n)} + 4\zeta^{(n)}\zeta_{zz}^{(n)} + 3(\zeta_z^{(n)})^2 + 6(\zeta^{(n)})^2 \zeta_z^{(n)} + (\zeta^{(n)})^4 - 20\varphi_z^{(n)}\varphi_{zz}^{(n)} + \\ & - 10(\varphi^{(n)})^2 (\zeta_z^{(n)} + (\zeta^{(n)})^2) - 20\varphi^{(n)}\varphi_z^{(n)}\zeta^{(n)} - 10(\varphi_z^{(n)})^2 + 30(\varphi^{(n)})^4 - z = 0. \end{aligned} \quad (13.7)$$

It is essential to find power expansions of its solutions. The carrier of the equation (13.7) consists of the points

$$\begin{aligned}
Q_{1,1} &= (-3, 1), \quad Q_{1,2} = (-2, 2), \quad Q_{1,3} = (-1, 3), \\
Q_{1,4} &= (0, 4), \quad Q_{1,5} = (1, 0), \quad Q_{2,k} = \left(\frac{1}{4} - \frac{5}{4}k, 0\right), \\
Q_{3,k} &= \left(-\frac{1}{2} - \frac{5}{4}k, 1\right), \quad Q_{4,k} = \left(\frac{1}{2} - \frac{5}{4}k, 2\right), \quad Q_{5,k} = \left(-\frac{3}{2} - \frac{5}{4}k, 0\right), \\
Q_{6,k} &= \left(1 - \frac{5}{4}k, 0\right), \quad k \in \mathbb{N} \cup \{0\},
\end{aligned} \tag{13.8}$$

where the doubled index in the points numeration is introduced for the notational convenience. The convex closing of the points (13.8) is the half-string presented at the fig. 7. The periphery of the half-string contains two

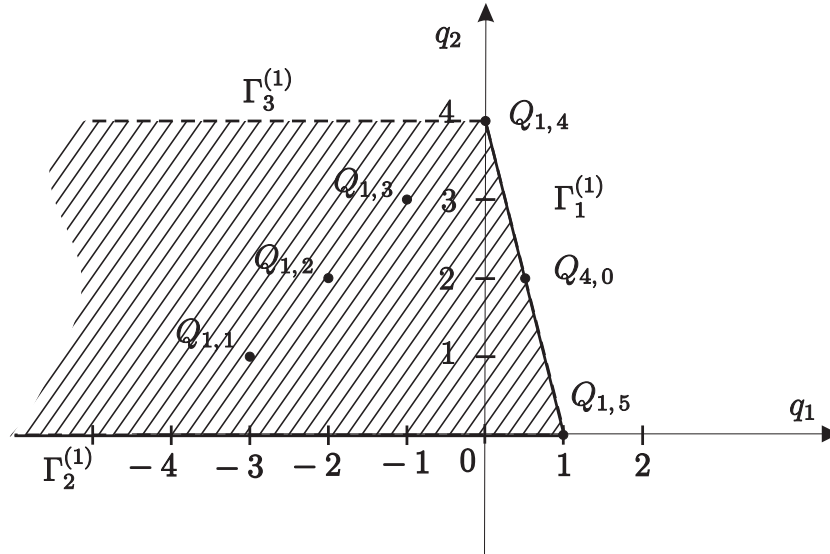


Fig. 7.

apexes $Q_{1,4}, Q_{1,5}$ and three edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3$) with the normal vectors $N_1 = (4, 1)$, $N_2 = (0, -1)$, $N_3 = (0, 1)$. Suitable expansions can be obtained examining the reduced equation corresponding to the edge $\Gamma_1^{(1)}$

$$h_1^{(1)}(z, \zeta^{(n)}) \stackrel{def}{=} (\zeta^{(n)})^4 - 10(c_{1/4}^{(n)})^2(\zeta^{(n)})^2 z^{\frac{1}{2}} + (30(c_{1/4}^{(n)})^4 - 1)z = 0. \tag{13.9}$$

It has sixteen solutions

$$\zeta^{(n,l)}(z) = g_{1/4}^{(n,l)} z^{1/4}, \quad n, l = 1, 2, 3, 4, \tag{13.10}$$

where

$$g_{1/4}^{(n; l=\{1,2,3,4\})} = \pm \left(5(c_{1/4}^{(n)})^2 \pm \sqrt{1 - 5(c_{1/4}^{(n)})^4} \right)^{1/2}.$$

Taking into account the fact that $(c_{1/4}^{(n)})^4 = 1/6$, we get

$$\begin{aligned} g_{1/4}^{(n; l=\{1,2\})} &= \pm 6^{1/4}, n = 1, 2 ; g_{1/4}^{(n; l=\{3,4\})} = \pm (8/3)^{1/4}, n = 1, 2; \\ g_{1/4}^{(n; l=\{1,2\})} &= \pm 6^{1/4}i, n = 3, 4 ; g_{1/4}^{(n; l=\{3,4\})} = \pm (8/3)^{1/4}i, n = 3, 4. \end{aligned}$$

The reduced equation is an algebraic one that is why it does not have critical numbers. Now we will look for the power expansions of the equation (13.7) solutions which possess the power asymptotic (13.10). The shifted carrier of the equation (13.7) lies in the lattice generated by the vectors $B_7 = (5/4, 0)$, $B_8 = (1, 1)$. The shifted carrier of the reduced solutions (13.10) gives the vector $B_9 = (-1/4, 1)$. Taking into consideration the correlation $B_8 - B_9 = (5/4, 0) = B_7$ we get that the vector B_9 belongs to the lattice with the basis B_7, B_8 . The points of this lattice are the following

$$Q = (q_1, q_2) = k(1, 1) + m \left(\frac{5}{4}, 0 \right) = \left(k + \frac{5m}{4}, k \right).$$

At the line $q_2 = -1$ we have $k = -1$ and hence $q_1 = -1 + 5m/4$. Since the cone of the problem is $\mathcal{K} = \left\{ k < \frac{1}{4} \right\}$, then the set \mathbf{K} can be presented in the form

$$\mathbf{K} = \left\{ \frac{1 - 5k}{4}, k \in \mathbb{N} \right\}. \quad (13.11)$$

Power expansions of the equation (13.7) solutions are

$$\zeta^{(n,l)}(z) = g_{1/4}^{(n,l)} z^{1/4} + \sum_{k \in \mathbb{N}} g_{(1-5k)/4}^{(n,l)} z^{(1-5k)/4}, \quad n, l = 1, 2, 3, 4. \quad (13.12)$$

Using the values of the coefficients $c_{1/4}^{(n)}$, $c_{-1}^{(n)}$ and $g_{1/4}^{(n,l)}$ we can compute the coefficient $g_{-1}^{(n,l)}$.

$$\begin{aligned} g_{-1}^{(n,l)} &= -\frac{3}{8}, n = 1, 2, 3, 4, l = 1, 2; \quad g_{-1}^{(n,l)} = -\frac{3}{8} + \frac{5}{4}\alpha, (n, l) = \{(1, 3), (2, 4), (3, 3), (4, 4)\}; \\ g_{-1}^{(n,l)} &= -\frac{3}{8} - \frac{5}{4}\alpha, (n, l) = \{(1, 4), (2, 3), (3, 4), (4, 3)\}. \end{aligned}$$

In view of the transformation (13.6) the additions $u^{(n,l)}(z)$ can be found. They are

$$u^{(n,l)}(z) = C_1 \exp \int \zeta^{(n,l)}(z) dz. \quad (13.13)$$

Where from we get

$$\begin{aligned} u^{(n,l)}(z) &= C_1 z^{g_{-1}^{(n,l)}} \exp \left[\frac{4}{5} g_{1/4}^{(n,l)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} g_{(1-5k)/4}^{(n,l)} z^{5(1-k)/4} \right], \\ &n, l = 1, 2, 3, 4. \end{aligned} \quad (13.14)$$

Here C_1 and later C_2 and C_3 are the constants of integration. The additions $u^{(n,l)}(z)$ are exponentially small at $z \rightarrow \infty$ in those sectors of the complex plane z , where

$$\operatorname{Re} \left[g_{1/4}^{(n,l)} z^{5/4} \right] < 0. \quad (13.15)$$

Consequently for each expansion $G_3^{(1)} n$ we have found four one-parametric families of additions $G_2^{(1)} n G_1^{(1)} l$ ($n, l = 1, 2, 3, 4$).

14 Exponential additions of the second level, corresponding to the edge $\Gamma_2^{(1)}$ at $\alpha \neq 0$.

In this section we will compute the exponential additions of the second level $v^{(n,l)}(z)$, i.e. the additions to the solutions $\zeta^{(n,l)}(z)$. The reduced equation to the addition $v^{(n,l)}(z)$ is

$$M_{n,l}^{(2)}(z) v^{(n,l)}(z) = 0, \quad (14.1)$$

where an operator $M_{n,l}^{(2)}(z)$ is the first variation of (13.7) at the solutions $\zeta^{(n,l)}(z)$. The equation (14.1) can be rewritten as

$$\frac{d^3 v}{dz^3} + 4\zeta \frac{d^2 v}{dz^2} + 2(3\zeta_z + 3\zeta^2 - 5\varphi^2) \frac{dv}{dz} + 4(\zeta_{zz} + 3\zeta\zeta_z + \zeta^3 - 5\varphi^2\zeta - 5\varphi\varphi_z)v = 0. \quad (14.2)$$

Here and later $\varphi \stackrel{\text{def}}{=} \varphi^{(n)}$, $v \stackrel{\text{def}}{=} v^{(n,l)}$, $\zeta \stackrel{\text{def}}{=} \zeta^{(n,l)}$. Making the transformation of the variables

$$\frac{d \ln v}{dz} = \xi, \quad (14.3)$$

we get

$$\frac{dv}{dz} = \xi v, \quad \frac{d^2 v}{dz^2} = \xi_z v + \xi^2 v, \quad \frac{d^3 v}{dz^3} = \xi_{zz} v + 3\xi \xi_z v + \zeta^3 v.$$

And the equation (14.2) transferees to

$$\begin{aligned} \xi_{zz} + 3\xi \xi_z + \xi^3 + 4\zeta(\xi_z + \xi^2) + 6(\zeta_z + \zeta^2)\xi - 10\varphi^2\xi + 4\zeta_{zz} + 12\zeta\zeta_z + \\ + 4\zeta^3 - 20\varphi^2\zeta - 20\varphi\varphi_z = 0. \end{aligned} \quad (14.4)$$

The carrier of this equation is defined by the following set of points

$$\begin{aligned} M_{1,1} = (-2, 1), \quad M_{1,2} = (-1, 2), \quad M_{1,3} = (0, 3), \quad M_{2,k} = \left(-\frac{3}{4} - \frac{5}{4}k, 1\right), \\ M_{3,k} = \left(\frac{1}{4} - \frac{5}{4}k, 2\right), \quad M_{4,k} = \left(\frac{1}{2} - \frac{5}{4}k, 1\right), \quad M_{5,k} = \left(-\frac{7}{4} - \frac{5}{4}k, 0\right), \\ M_{6,k} = \left(-\frac{1}{2} - \frac{5}{4}k, 0\right), \quad M_{7,k} = \left(\frac{3}{4} - \frac{5}{4}k, 0\right), \quad k \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (14.5)$$

Their convex hull is the half-string similar to the one presented at the fig. 7. Its edges $\Gamma_3^{(1)}$, $\Gamma_2^{(1)}$ are the rays starting out of the points $(0, 3)$, $(3/4, 0)$, accordingly, and the edge $\Gamma_1^{(1)}$ connects the points $(3/4, 0)$ $(0, 3)$. Examining the edge $\Gamma_1^{(1)}$ which contains the points $(0, 3)$, $(1/4, 2)$, $(1/2, 1)$ and $(3/4, 0)$ we can find suitable solutions. The reduced equation corresponding to this edge is

$$\begin{aligned} \xi^3 + 4g_{1/4}^{(n,l)} z^{1/4} \xi^2 + 2[3(g_{1/4}^{(n,l)})^2 - 5(c_{1/4}^{(n)})^2] z^{1/2} \xi + \\ + 4g_{1/4}^{(n,l)} [(g_{1/4}^{(n,l)})^2 - 5(c_{1/4}^{(n)})^2] z^{3/4} = 0. \end{aligned} \quad (14.6)$$

The solutions of the equation (14.6) can be presented in the form

$$\xi^{(n,l,m)}(z) = r_{1/4}^{(n,l,m)} z^{1/4}, \quad n, l = 1, 2, 3, 4; \quad m = 1, 2, 3, \quad (14.7)$$

where $r \stackrel{\text{def}}{=} r_{1/4}^{(n,l,m)}$ are the roots of the cubic equation

$$\begin{aligned} r^3 + 4g_{1/4}^{(n,l)} r^2 + 2(3(g_{1/4}^{(n,l)})^2 - 5(c_{1/4}^{(n)})^2) r + \\ + 4(g_{1/4}^{(n,l)})^3 - 20g_{1/4}^{(n,l)} (c_{1/4}^{(n)})^2 = 0. \end{aligned} \quad (14.8)$$

Solving this equation we get

$$\begin{aligned} r_{1/4}^{(n,l,1)} = -2g_{1/4}^{(n,l)}, \quad r_{1/4}^{(n,l,2)} = -g_{1/4}^{(n,l)} + \left(10(c_{1/4}^{(n)})^2 - (g_{1/4}^{(n,l)})^2\right)^{1/2} \\ r_{1/4}^{(n,l,3)} = -g_{1/4}^{(n,l)} - \left(10(c_{1/4}^{(n)})^2 - (g_{1/4}^{(n,l)})^2\right)^{1/2}. \end{aligned}$$

The vectors $(1, 1)$, $(5/4, 0)$ are the basis of the lattice which corresponds to the shifted carrier of the equation (14.4). The shifted carrier $(-1/4, 1)$ of the reduced solutions (14.7) belongs to this lattice. Consequently the set \mathbf{K} coincides with (13.11). Power expansions of the functions $\xi^{(n,l,m)}(z)$ are the following

$$\begin{aligned} \xi^{(n,l,m)}(z) = r_{1/4}^{(n,l,m)} z^{1/4} + \sum_{k \in \mathbb{N}} r_{(1-5k)/4}^{(n,l,m)} z^{(1-5k)/4}, \\ n, l = 1, 2, 3, 4; \quad m = 1, 2, 3. \end{aligned} \quad (14.9)$$

Calculation of the coefficients $r_{-1}^{(n,l,m)}$ yields $r_{-1}^{(n,l,m)} = 1/4$ at $n = 1, 2, 3, 4; \quad l = 1, 2; \quad m = 1$. In other cases $r_{-1}^{(n,l,m)}$ depend on the parameter α .

So we have found the exponential additions $v^{(n,l,m)}(z)$ to the solutions $\zeta^{(n,l)}(z)$

$$\begin{aligned} v^{(n,l,m)}(z) = C_2 z^{r_{-1}^{(n,l,m)}} \exp \left[\frac{4}{5} r_{1/4}^{(n,l,m)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} r_{(1-5k)/4}^{(n,l,m)} z^{5(1-k)/4} \right] \\ n = 1, 2, 3, 4; \quad l = 1, 2, 3, 4; \quad m = 1, 2, 3. \end{aligned} \quad (14.10)$$

They are exponentially small provided that

$$\text{Re} \left[r_{1/4}^{(n,l,m)} z^{5/4} \right] < 0, \quad z \rightarrow \infty. \quad (14.11)$$

Thus, taking into account two-level additions $G_2^{(1)}n G_1^{(1)}l G_1^{(1)}m$, we have 48 expansions of the equation (1.3) solutions $(n, l = 1, 2, 3, 4; m = 1, 2, 3)$.

15 Exponential additions of the third level, corresponding to the edge $\Gamma_2^{(1)}$ at $\alpha \neq 0$.

In this section we will look for exponential additions of the third level $y^{(n,l,m)}(z)$, i.e. we will look for additions to the solutions $\xi^{(n,l,m)}(z)$. The reduced equation to the addition $y^{(n,l,m)}(z)$ is the following

$$M_{n,l,m}^{(3)}(z)y^{(n,l,m)} = 0. \quad (15.1)$$

Operator $M_{n,l,m}^{(3)}(z)$ can be found as the first variation of (14.4) at the solutions $\xi^{(n,l,m)}(z)$ and then the equation (15.1) for the function $y \stackrel{\text{def}}{=} y^{(n,l,m)}$ can be rewritten as

$$y_{zz} + (3\xi + 4\zeta)y_z + (3\xi_z + 3\xi^2 + 8\xi\zeta + 6\zeta_z + 6\zeta^2 - 10\varphi^2)y = 0. \quad (15.2)$$

Introducing the new variable η by the rule

$$\frac{d \ln y}{dz} = \eta \quad (15.3)$$

we get that

$$\frac{dy}{dz} = \eta y, \quad \frac{d^2 y}{dz^2} = \eta_z y + \eta^2 y.$$

Hence equation (15.2) transfers to the following

$$\eta_z + \eta^2 + (3\xi + 4\zeta)\eta + 3\xi_z + 3\xi^2 + 8\xi\zeta + 6\zeta_z + 6\zeta^2 - 10\varphi^2 = 0. \quad (15.4)$$

The carrier of this equation is composed of the points

$$\begin{aligned} R_{1,1} &= (-1, 1), \quad R_{1,2} = (0, 2), \quad R_{2,k} = \left(\frac{1}{4} - \frac{5}{4}k, 1\right), \quad R_{3,k} = \left(-\frac{3}{4} - \frac{5}{4}k, 0\right), \\ R_{4,k} &= \left(\frac{1}{2} - \frac{5}{4}k, 0\right), \quad k \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (15.5)$$

Closing the convex hull based on these points we obtain the half-string similar to the one presented at fig. 7. Its edges $\Gamma_3^{(1)}$ and $\Gamma_2^{(1)}$ are the rays starting out of the points $(0, 2)$ and $(1/2, 0)$, accordingly. The edge $\Gamma_1^{(1)}$ is limited by the points $(0, 2)$ and $(1/2, 0)$. Now we will examine the reduced equation

$$\begin{aligned} \eta^2 + (3r_{1/4}^{(n,l,m)} + 4g_{1/4}^{(n,l)})z^{1/4}\eta + (3(r_{1/4}^{(n,l,m)})^2 + 8r_{1/4}^{(n,l,m)}g_{1/4}^{(n,l)} + \\ 6(g_{1/4}^{(n,l)})^2 - 10(c_{1/4}^{(n)})^2)z^{1/2} = 0, \end{aligned} \quad (15.6)$$

which corresponds to the edge $\Gamma_1^{(1)}$ (this edge contains three points $(0, 2)$, $(1/4, 1)$ and $(1/2, 0)$). The equation (15.6) possesses the following solutions

$$\eta^{(n,l,m,p)}(z) = q_{1/4}^{(n,l,m,p)} z^{1/4}, \quad n, l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2 \quad (15.7)$$

where $q \stackrel{def}{=} q_{1/4}^{(n,l,m,p)}$ are the solutions of the quadratic equation

$$\begin{aligned} q^2 + (3r_{1/4}^{(n,l,m)} + 4g_{1/4}^{(n,l)})q + 3(r_{1/4}^{(n,l,m)})^2 + 8r_{1/4}^{(n,l,m)}g_{1/4}^{(n,l)} + \\ 6(g_{1/4}^{(n,l)})^2 - 10(c_{1/4}^{(n)})^2 = 0. \end{aligned} \quad (15.8)$$

At fixed n, l, m it has two solutions

$$\begin{aligned} q_{1/4}^{(n,l,m,p=\{1,2\})} = -\frac{3}{2}r_{1/4}^{(n,l,m)} - 2g_{1/4}^{(n,l)} \pm \\ \pm \frac{1}{2} \left(40(c_{1/4}^{(n)})^2 - 3(r_{1/4}^{(n,l,m)})^2 - 8r_{1/4}^{(n,l,m)}g_{1/4}^{(n,l)} - 8(g_{1/4}^{(n,l)})^2 \right)^{(1/2)}. \end{aligned}$$

The basis of the lattice defined by the shifted carrier of the equation (15.4) is composed of the vectors $(1, 1)$, $(5/4, 0)$. The set \mathbf{K} coincides with (13.11). Power expansions for $\eta^{(n,l,m,p)}(z)$ can be presented as

$$\begin{aligned} \eta^{(n,l,m,p)}(z) = q_{1/4}^{(n,l,m,p)} z^{1/4} + \sum_{k \in \mathbb{N}} q_{(1-5k)/4}^{(n,l,m,p)} z^{(1-5k)/4}, \\ n, l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2. \end{aligned} \quad (15.9)$$

It can be shown that the coefficients $q_{-1}^{(n,l,m,p)}$ depend on α . The exponential additions $y^{(n,l,m,p)}(z)$ to the solutions $\xi^{(n,l,m)}(z)$ can be written as

$$\begin{aligned} y^{(n,l,m,p)}(z) = C_3 z^{q_{-1}^{(n,l,m,p)}} \exp \left[\frac{4}{5} q_{1/4}^{(n,l,m,p)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} q_{(1-5k)/4}^{(n,l,m,p)} z^{5(1-k)/4} \right], \\ n, l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2. \end{aligned} \quad (15.10)$$

They are exponentially small when

$$\text{Re} \left[q_{1/4}^{(n,l,m,p)} z^{5/4} \right] < 0, \quad z \rightarrow \infty. \quad (15.11)$$

Thus for the solution expansions of the studied equation near the point $z = \infty$ three-level exponential additions have been found. Taking into account exponential additions, the solutions $w(z)$ at $z \rightarrow \infty$ can be written as

$$\begin{aligned} w(z) = \varphi^{(n)}(z) + \exp \left[\int dz \left(\zeta^{(n,l)}(z) + \exp \left[\int dz \left(\xi^{(n,l,m)}(z) + \right. \right. \right. \right. \\ \left. \left. \left. \exp \left(\int dz \eta^{(n,l,m,p)}(z) \right) \right) \right] \right) \right] \end{aligned}$$

or as

$$\begin{aligned}
w(z) = & \varphi^{(n)}(z) + \exp \left[u^{(n,l)}(z) + \int dz \exp \left[v^{(n,l,m)}(z) + \right. \right. \\
& \left. \left. + \int dz \exp \left(y^{(n,l,m,p)}(z) \right) \right] \right], \\
& n, l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2.
\end{aligned} \tag{15.12}$$

Denote these solutions (in view of three-level additions) as $G_2^{(1)} n G_1^{(1)} l G_1^{(1)} m G_1^{(1)} p$ where $n, l = 1, 2, 3, 4$; $m = 1, 2, 3$; $p = 1, 2$.

16 Exponential additions of the first level, corresponding to the edge $\Gamma_3^{(1)}$ at $\alpha \neq 0$.

Let us find the exponential addition of the first level $u(z)$ to the expansion (6.5), i.e. we will look for the solutions in the form

$$w(z) = \psi(z) + u(z). \tag{16.1}$$

Everything written in section 13 up to the formula (13.7) is true in the case of the edge $\Gamma_3^{(1)}$, taking into account the only fact that instead of four expansions $\varphi^{(n)}(z)$ we have only one $\psi(z)$. Thus we obtain the equation

$$\begin{aligned}
& \zeta_{zzz} + 4\zeta\zeta_{zz} + 3(\zeta_z)^2 + 6\zeta^2\zeta_z + \zeta^4 - 10\psi^2(\zeta_z + \zeta^2) - \\
& - 20\psi\psi_z\zeta - 20\psi_z\psi_{zz} - 10(\psi_z)^2 + 30\psi^4 - z = 0.
\end{aligned} \tag{16.2}$$

Its carrier consists of the points

$$\begin{aligned}
Q_{1,1} &= (-3, 1), \quad Q_{1,2} = (-2, 2), \quad Q_{1,3} = (-1, 3), \quad Q_{1,4} = (0, 4), \\
Q_{1,5} &= (1, 0), \quad Q_{2,k} = (-3 - 5k, 1), \quad Q_{3,k} = (-2 - 5k, 2), \\
Q_{4,k} &= (-4 - 5k, 0), \quad k \in \mathbb{N} \cup \{0\}.
\end{aligned} \tag{16.3}$$

Convex closing of these points yields the half-string presented at fig. 8. Its periphery is composed of two apexes $Q_{1,4}$, $Q_{1,5}$ and of three edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3$) with the normal vectors $N_1 = (4, 1)$, $N_2 = (0, -1)$, $N_3 = (0, 1)$. The sufficient solutions are given by the edge $\Gamma_1^{(1)}$. It is characterized by the following reduced equation

$$h_1^{(1)}(z, \zeta) \stackrel{def}{=} \zeta^4 - z = 0. \tag{16.4}$$

Equation (16.4) has four solutions

$$\zeta^{(l)}(z) = g_{1/4}^{(l)} z^{1/4}, \quad l = 1, 2, 3, 4; \quad g_{1/4}^{(l=\{1,2,3,4\})} = 1, -1, i, -i. \tag{16.5}$$

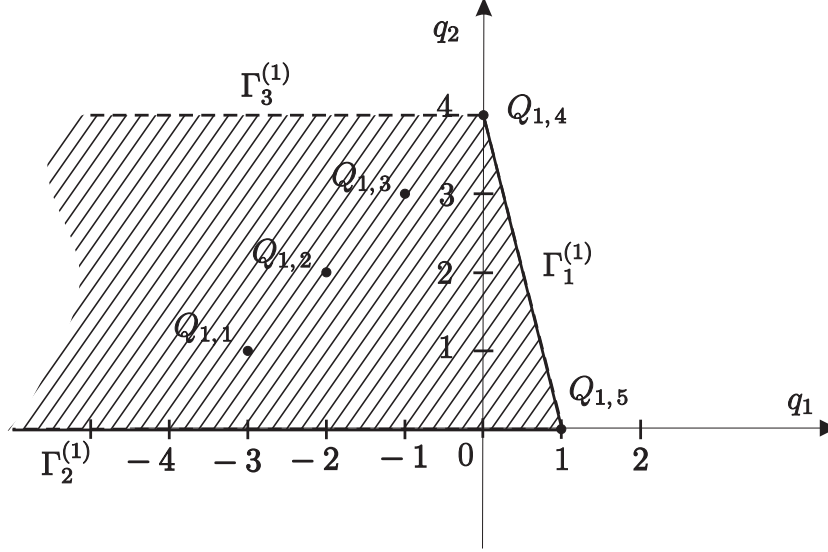


Fig. 8.

The shifted carrier of the equation (16.2) lies in the lattice with the basis $(-1, 4), (-2, 3)$. Together with the shifted carrier of the reduced solutions (16.5) they generate the new lattice with the basis $(-1/4, 1), (-2, 3)$. The cone of the problem is $\mathcal{K} = \{k < \frac{1}{4}\}$, then we get

$$\mathbf{K} = \left\{ \frac{1-5k}{4}, k \in \mathbb{N} \right\}. \quad (16.6)$$

Power expansions for the reduced solutions (16.5) can be written as

$$\zeta^{(l)}(z) = g_{1/4}^{(l)} z^{1/4} + \sum_{k \in \mathbb{N}} g_{(1-5k)/4}^{(l)} z^{(1-5k)/4}, \quad l = 1, 2, 3, 4. \quad (16.7)$$

In this expression $g_{-1}^{(l)} = -3/8$. Using the formula (13.13) we get exponential additions

$$u^{(l)}(z) = \frac{\tilde{C}_1}{z^{3/8}} \exp \left[\frac{4}{5} g_{1/4}^{(l)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} g_{(1-5k)/4}^{(l)} z^{5(1-k)/4} \right], \quad l = 1, 2, 3, 4. \quad (16.8)$$

Denote them as $G_3^{(1)} G_1^{(1)} l$. Here and later $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ are the arbitrary constants. Taking into consideration first two members of the series in (16.8) we can rewrite this expression in the form

$$u^{(l)}(z) = \frac{\tilde{C}_1}{z^{3/8}} \exp \left[\frac{4}{5} g_{1/4}^{(l)} z^{5/4} - \frac{64\alpha^2 - 9}{32g_{1/4}^{(l)} z^{5/4}} - \frac{5(64\alpha^2 - 9)}{256(g_{1/4}^{(l)})^2 z^{5/2}} + \dots \right], \quad l = 1, 2, 3, 4.$$

The applicability condition (13.15) holds for the additions (16.8) provided that $g_{1/4}^{(n,l)}$ replace $g_{1/4}^{(l)}$.

17 Exponential additions of the second level, corresponding to the edge $\Gamma_3^{(1)}$ at $\alpha \neq 0$.

Now we will look for exponential additions of the second level $v^{(l)}(z)$, i.e. we will look for the additions to the solutions $\zeta^{(l)}(z)$. All transformations of the section 14, which converted the equation (14.1) to the equation (14.4) do not change under the condition $v \stackrel{def}{=} v^{(l)} \quad \zeta \stackrel{def}{=} \zeta^{(l)}$. As a result we get the equation

$$\begin{aligned} \xi_{zz} + 3\xi\xi_z + \xi^3 + 4\zeta(\xi_z + \xi^2) + 6(\zeta_z + \zeta^2)\xi - 10\psi^2\xi + 4\zeta_{zz} + 12\zeta\zeta_z + \\ + 4\zeta^3 - 20\psi^2\zeta - 20\psi\psi_z = 0. \end{aligned} \quad (17.1)$$

The carrier of this equation differs from the carrier of the equation (14.4):

$$\begin{aligned} M_{1,1} = (-2, 1), \quad M_{1,2} = (-1, 2), \quad M_{1,3} = (0, 3), \quad M_{2,k_1} = \left(-\frac{3}{4} - \frac{5}{4}k_1, 1\right), \\ M_{3,k_1} = \left(\frac{1}{4} - \frac{5}{4}k_1, 2\right), \quad M_{4,k_1} = \left(\frac{1}{2} - \frac{5}{4}k_1, 1\right), \quad M_{5,k_1} = (-2 - 5k_1, 1), \\ M_{6,k_1} = \left(-\frac{7}{4} - \frac{5}{4}k_1, 0\right), \quad M_{7,k_1} = \left(-\frac{1}{2} - \frac{5}{4}k_1, 0\right), \quad M_{8,k_1} = \left(\frac{3}{4} - \frac{5}{4}k_1, 0\right), \\ M_{9,k_1} = (-3 - 5k_1, 0), \quad M_{10,k_1,k_2} = \left(-\frac{7}{4} - 5k_1 - \frac{5}{4}k_2, 0\right), \quad k_i \in \mathbb{N} \cup \{0\}, \quad i = 1, 2. \end{aligned} \quad (17.2)$$

The convex hull obtained after locking of these points is similar to the one described in the section 14. The reduced equation corresponding to the edge $\Gamma_1^{(1)}$ is the following

$$\xi^3 + 4g_{1/4}^{(l)}z^{1/4}\xi^2 + 6(g_{1/4}^{(l)})^2z^{1/2}\xi + 4(g_{1/4}^{(l)})^3z^{3/4} = 0. \quad (17.3)$$

The equation (17.3) has 12 solutions

$$\xi^{(l,m)}(z) = r_{1/4}^{(l,m)}z^{1/4}, \quad l = 1, 2, 3, 4; \quad m = 1, 2, 3, \quad (17.4)$$

where calculation of the coefficients $r_{1/4}^{(l,m)}$ yields

$$r_{1/4}^{(l,1)} = -2g_{1/4}^{(l)}; \quad r_{1/4}^{(l,2)} = (-1+i)g_{1/4}^{(l)}; \quad r_{1/4}^{(l,3)} = -(1+i)g_{1/4}^{(l)}, \quad l = 1, 2, 3, 4.$$

The lattice which contains the shifted carriers of the equation (17.1) and the reduced solutions (17.4) coincides with the lattice of the section 13. That is why power expansions for the reduced solutions (17.4) are

$$\xi^{(l,m)} = r_{1/4}^{(l,m)}z^{1/4} + \sum_{k \in \mathbb{N}} r_{(1-5k)/4}^{(l,m)}z^{(1-5k)/4}, \quad l = 1, 2, 3, 4; \quad m = 1, 2, 3. \quad (17.5)$$

Here $r_{-1}^{(l,m)} = 1/4$. Returning to the variables $v^{(l,m)}(z)$ we get the exponential additions $G_3^{(1)} G_1^{(1)} l G_1^{(1)} m$ and their applicability condition

$$v^{(l,m)}(z) = \tilde{C}_2 z^{1/4} \exp \left[\frac{4}{5} r_{1/4}^{(l,m)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} r_{(1-5k)/4}^{(l,m)} z^{5(1-k)/4} \right]; \quad (17.6)$$

$$\operatorname{Re} \left[r_{1/4}^{(l,m)} z^{5/4} \right] < 0, \quad z \rightarrow \infty; \quad l = 1, 2, 3, 4; \quad m = 1, 2, 3.$$

18 Exponential additions of the third level, corresponding to the edge $\Gamma_3^{(1)}$ at $\alpha \neq 0$.

Exponential additions of the third level $y^{(l,m)}(z)$ (additions to the solutions $\xi^{(l,m)}(z)$) can be found as they have been found for the edge $\Gamma_2^{(1)}$ in the section 15. Let us in brief follow the procedure of calculations. After essential transformations we get the equation

$$\eta_z + \eta^2 + (3\xi + 4\zeta)\eta + 3\xi_z + 3\xi^2 + 8\xi\zeta + 6\zeta_z + 6\zeta^2 - 10\psi^2 = 0. \quad (18.1)$$

Its carrier consists of the following points

$$R_{1,1} = (-1, 1), \quad R_{1,2} = (0, 2), \quad R_{2,k} = \left(-\frac{3}{4} - \frac{5}{4}k, 0\right), \quad R_{3,k} = \left(\frac{1}{2} - \frac{5}{4}k, 0\right),$$

$$R_{4,k} = (-2 - 5k, 0), \quad R_{5,k} = \left(\frac{1}{4} - \frac{5}{4}k, 1\right), \quad k \in \mathbb{N} \cup \{0\}. \quad (18.2)$$

Closing of their convex hull yields the half-string similar to the one of the section 15. The edge $\Gamma_1^{(1)}$ is characterized by the reduced equation

$$\eta^2 + (3r_{1/4}^{(l,m)} + 4g_{1/4}^{(l)})z^{1/4}\eta + \left(3(r_{1/4}^{(l,m)})^2 + 8r_{1/4}^{(l,m)}g_{1/4}^{(l)} + 6(g_{1/4}^{(l)})^2\right)z^{1/2} = 0 \quad (18.3)$$

which possesses 24 solutions

$$\eta^{(l,m,p)}(z) = q_{1/4}^{(l,m,p)} z^{1/4}; \quad l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2. \quad (18.4)$$

The coefficients $q_{1/4}^{(l,m,p)}$ are equal to

$$q_{1/4}^{(l,m,p=\{1,2\})} = -\frac{3}{2} r_{1/4}^{(l,m)} - 2 g_{1/4}^{(l)} \pm \frac{1}{2} i \sqrt{(3(r_{1/4}^{(l,m)})^2 + 8r_{1/4}^{(l,m)}g_{1/4}^{(l)} + 8(g_{1/4}^{(l)})^2)},$$

$$l = 1, 2, 3, 4; \quad m = 1, 2, 3.$$

The set \mathbf{K} in this case is (13.11). Consequently power expansions for the

solutions (18.4) can be written as

$$\eta^{(l,m,p)}(z) = q_{1/4}^{(l,m,p)} z^{1/4} + \sum_{k \in \mathbb{N}} q_{(1-5k)/4}^{(l,m,p)} z^{(1-5k)/4}, \quad (18.5)$$

$$l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2.$$

The coefficients $q_{-1}^{(l,m,p)}$ do not depend on α and $q_{-1}^{(l,m,p)} = 1/4$.

So the exponential additions $G_3^{(1)} G_1^{(1)} l G_1^{(1)} m G_1^{(1)} p$ are the following

$$y^{(l,m,p)}(z) = \tilde{C}_3 z^{1/4} \exp \left[\frac{4}{5} q_{1/4}^{(l,m,p)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} q_{(1-5k)/4}^{(l,m,p)} z^{5(1-k)/4} \right],$$

$$l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2. \quad (18.6)$$

The applicability condition for the additions (18.6) coincides with (15.11) provided that $q_{1/4}^{(n,l,m,p)}$ replace $q_{1/4}^{(l,m,p)}$. Taking into account obtained additions the solution $w(z)$ can be written as it has been done in formula (15.12).

19 Conclusion.

In this work the fourth-order analogue to the second Painlevé equation was studied with a help of the power geometry method [42–44]. We found all power and non-power asymptotics of its solutions, power expansions generated by these power asymptotics and exponential additions which correspond to certain expansions. The results depend on the value of parameter α .

First of all let us briefly review the case $\alpha \neq 0$. Near the point $z = 0$ we obtained one-parametric family $G_1^{(0)} 4$, three two-parametric families $G_1^{(0)} 3$, $G_1^{(1)} 3$, $G_1^{(1)} 4$, three three-parametric families $G_1^{(0)} 2$, $G_1^{(1)} 1$, $G_1^{(1)} 2$, one four-parametric family $G_1^{(0)} 1$ and the family $G_4^{(1)} 1$ (the families $G_1^{(0)} 2$, $G_1^{(0)} 3$, $G_1^{(0)} 4$, $G_4^{(1)} 1$ are the special cases of $G_1^{(0)} 1$). These expansions converge for small $|z|$. Their existence and analyticity follow from the Cauchy theorem.

Besides near the point $z = \infty$ we found the families: $G_2^{(1)} n$ ($n = 1, 2, 3, 4$) and $G_3^{(1)} 1$. For each of these expansions three-level exponential additions we computed: $G_2^{(1)} n G_1^{(1)} l G_1^{(1)} m G_1^{(1)} p$ ($n, l = 1, 2, 3, 4$; $m = 1, 2, 3$; $p = 1, 2$) and $G_3^{(1)} G_1^{(1)} l G_1^{(1)} m G_1^{(1)} p$ ($l = 1, 2, 3, 4$; $m = 1, 2, 3$; $p = 1, 2$).

In addition it is important to mention that some partial solutions of the studied equation can be found using the expansion $G_3^{(1)} 1$. It is obvious that at the certain values of the parameter α (more exactly $\alpha = -2, -1, 1, 2$) the series

in the formula (6.5) truncates and we get rational solutions

$$w(z) = \frac{-\alpha}{z}, \quad \alpha = -2, -1, 1, 2. \quad (19.1)$$

Application of these solutions as "seed solutions" in the Bäcklund transformations for the studied equation yields other rational solutions at whole values of the parameter α . Partial solutions (19.1) can be also obtained from the expansions $G_1^{(1)}1$, $G_1^{(1)}2$, $G_1^{(1)}3$ and $G_1^{(1)}4$.

In the case $\alpha = 0$ near the point $z = 0$ we found one-parametric family $\tilde{G}_1^{(0)}4$, three two-parametric families $\tilde{G}_1^{(0)}3$, $\tilde{G}_1^{(1)}3$, $\tilde{G}_1^{(1)}4$, three three-parametric families $\tilde{G}_1^{(0)}2$, $\tilde{G}_1^{(1)}1$, $\tilde{G}_1^{(1)}2$ and one four-parametric family $\tilde{G}_1^{(0)}1$ of power expansions (the families $\tilde{G}_1^{(0)}2$, $\tilde{G}_1^{(0)}3$, $\tilde{G}_1^{(0)}4$ are the special cases of $G_1^{(0)}1$). All of them converge for small $|z|$.

Near the point $z = \infty$ we computed four families of non-power asymptotics $\tilde{G}_3^{(1)}n$, ($n = 1, 2, 3, 4$) and four families of power expansions $\tilde{G}_2^{(1)}n$ ($n = 1, 2, 3, 4$).

The expansions $G_1^{(1)}n$ ($n = 1, 2, 3, 4$) were found earlier [8, 17], while all other expansions are the new ones.

To sum up we would like to emphasize that the obtained power expansions differ from the power expansions of the Painlevé equations $P_1 \div P_6$ solutions [45–49]. This fact can be interpreted as the additional proof of the hypothesis that the fourth-order equation (1.3) determines new transcendental functions as the equations $P_1 \div P_6$ do.

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